International Journal of Applied Mathematics
& Statistical Sciences (IJAMSS)
ISSN(P): 2319-3972; ISSN(E): 2319-3980
Vol. 5, Issue 2, Feb - Mar 2016; 125-146

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

FULLY AND NATURALLY CANCELLATION FUZZY MODULE

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ABSTRACT

Let R be a commutative ring with identity and let M be a unital R-module. Fully cancellation fuzzy modules and naturally cancellation fuzzy modules are characterized. Furthermore, some basic properties and some previous results on these concepts are introduced.

KEYWORDS: Commutative with Identity and All Modules

INTRODUCTION

Throughout this paper all ring are commutative with identity and all modules are unitary. Also we consider R to be a ring and M a unitary R-module. An R-module M is called a cancellation module if IN = IM, where I and J are ideal of R, then I = J [1, Definition (1.1)]

Recall that an R-module M is said to be a fully cancellation module if

IA = IB, where I is a non-zero ideal of R, A and B are sub modules of M [2, Definition (2.1)], and M is called naturally cancellation R-module if whenever K, N_1 and N_2 are sub modules of M such that $KN_1 = KN_2$, then$N_1 = N_2$ [2, Definition (2.2)].

In this paper, we fuzzily these concepts (Fully Cancellation and Naturally Cancellation) modules to fully cancellation fuzzy module and naturally cancellation fuzzy module

This paper consists of five sections in section one we give and recall many definitions and properties which will be needed to prove the results in the next sections.

In section two we introduce the definition of fully cancellation fuzzy module and we give some characterizations for a module to be fully cancellation fuzzy module. Also many properties and results of this concept are given.

In section three we introduce the definition of naturally of cancellation fuzzy module, many basic properties and results are studied.

In section four we study the relationships between fully cancellation and naturally cancellation fuzzy module.

In section five we discuss the direct sum of fully cancellation fuzzy module and many important results are presented.

1- PRELIMINARIES

This section contains some definitions and properties of fuzzy subsets, fuzzy ring, fuzzy ideal, fuzzy modules and fuzzy sub modules which will be used in the next sections.

Definition 1.1.1:

Let S be a non-empty set and I be the called interval [0, 1] of the real line (real numbers). A fuzzy set A is S (a fuzzy subset of S) is a function from S in to I. [3]

Definition 1.1.2

Let x_t: S $\longrightarrow [0,1]$ be a fuzzy set in S, where x∈S, t ∈[0, 1], define by x_t (y) =t if x=y and x_t (y) = 0 if x≠y, x_t is called a fuzzy singleton in S. [4]

Definition 1.1.3:

Let A and B be two fuzzy sets in S, then:

1-A=B if and only if A(x) = B(x), for all x∈S, [5]

2-A⊆B if and only if A(x) ≤ B(x), for all x∈S, [5]

3-A=B if and only if $A_t = B_t$, for all t∈[0,1], [3]

Proposition 1.1.4:

Let $a_{t,\, bk}$ be two fuzzy singletons of S. if $a_t = b_k$, then a=b and t=k, where t, k ∈[0, 1]. [6]

Definition1.1.5:

Let X and A be two fuzzy modules of R-module M. A is called a fuzzy sub module of X if A⊆X. [7]

Definition 1.1.6:

Let A be a fuzzy set in S, for all t ·∈[0, 1], the set A_t = {x∈S, A(x) ≥t} is called a level subset of A. [7]

Proposition 1.1.7:

A is a fuzzy sub module of fuzzy module X of an R- module M if and only if, A_t is a sub module of X_t, for each t∈[0,1]. [7]

Definition 1.1.8:

Let f be a mapping from a set M in to a set N, let A be a fuzzy set in M and B be a fuzzy set in N. The image of A denoted by f (A) is the fuzzy set in N defined by:

$$F(A)_{(y)} = \begin{cases} \sup\{A(z) | z \in f^{-1}(y), \text{if } f^{-1}(y) \neq \emptyset, \text{for all } y \in N \\ 0 \text{ otherwise} \end{cases}$$

And the inverse image of B denoted by f (B) is the fuzzy set in M defined by:

$f^{1}(B)(x)= B(f(x)$, for all x∈M.[3]

Definition 1.1.9:

Let M be an R-module. A fuzzy set X of M is called a fuzzy module of an R-module M if,

1-X (0) =1.

2-X(x-y) ≥min {X(x), X(y), for all x, y∈M.

3-X (rx)≥X(x), for all x∈M, r∈R.[4]

Proposition 1.1.10:

Let A and B be two fuzzy sub modules of fuzzy modules X and Y respectively, then

1-f (A) is a fuzzy sub module of Y.

2-f^{-1}(B) is a fuzzy sub module of X. [9]

Definition 1.1.11:

Let X and Y are fuzzy modules of R-modules M_1 and M_2 respectively, f: X⟶ Y

Is called fuzzy homomorphism if f: M_1⟶M_2 is R-homomorphism and Y (f(x)) =X(x) for each x∈M. [8]

Definition 1.1.12:

A fuzzy subset K of a ring R is called a fuzzy ideal of R, if for each x, y∈R:

1-K(x-y)≥ min {K(x), K(y)}

2-K (x y) ≥ max {K(x), K(y)} [10]

Definition 1.1.13:

Let X is a fuzzy module of an R-module M, let A be a fuzzy sub module of X and K be a fuzzy ideal of R, the product KA of K and A is defined by:

$$KA(x) = \begin{cases} \sup \{\inf \{k(r_1), \dots k(r_n), A(x_1), \dots\ A(x_n) \text{for some } r_i \in R, x_i \in M, n \in N \\ 0 \text{ otherwise} \end{cases}$$

Note that KA is a fuzzy sub module of X, [4] and $(KA)_t = K_t A_t$ for each t ∈[0, 1], [9].

Proposition 1.1.14:

A fuzzy subset K of R is a fuzzy ideal of R if and only if K_t, t ∈[0, 1] is an ideal of R. [10]

Proposition 1.1.15:

If X is a fuzzy module of an R-module M, then F-annX is a fuzzy ideal of R. [11]

Definition 1.1.16:

Let a be non- empty fuzzy sub module of a fuzzy module X. The fuzzy annihilator of a denoted by F-annA is denoted by:

(F-annA)(r) =sup {t: t ∈[0, 1], r_t. A⊆0_1}, for all r∈R

Note; F-annA = (0_1: A), hence $(F\text{-}annX)_t$⊆ annX_t. [4]

Definition 1.1.17:

Let A and B be two fuzzy sub modules of an R-module M. The addition A+B defined by: (A+B) (x) =sup {in f {A(y), B (z)} x=y + z, for all x, y, z∈M}. [4]

Definition 1.1.18:

Let A and B be two fuzzy sub modules of a fuzzy module X. The residual quotient of A and B denoted by (A:B) is the fuzzy subset of R defined by:

(A: B) (r) =sup {t∈ [0,1]: r_t. B ⊆ A}, for all r ∈R. that (A: B) = {r_t: r_tB⊆A; r_t is a fuzzy singleton of R}. If B=<x_k>, then (A: <$_k$>) = {r_t: rt x_k⊆A, r_t is of fuzzy singleton of R {. [4]

Remark 1.1.19:

If X is a fuzzy module of an R-module M and x_t⊆X, then for all fuzzy singleton r_k of R, $r_k x_t=(rx)_\lambda$, whereλ =min {k, t}. [11]

Proposition 1.1.20:

Let X be a fuzzy module of an R-module M, A be a fuzzy sub module of X and r_t be a fuzzy singleton of R, then r_t ∘A=<r_t>∘A from [2], r_t∘ A=r_t A. Then r_tA=<r_t>A. [11]

Definition 1.1.21:

Let A and B be two fuzzy sub modules of a fuzzy module X of an R-module M. Then (A: B) is a fuzzy ideal of R. [4]

2. FULLY CANCELLATION FUZZY MODULE

An R-module M is called fully cancellation module if for every non zero ideal I of R and for every sub modules N, W of M such that IN = IW, then N = W. [2, Definition (2.1)]

We shall fuzzify this concept to a fully cancellation fuzzy module.

Definition1.2.1:

Let X be a fuzzy module of an R-module M, X is called fully cancellation fuzzy module if for every non empty fuzzy ideal I of R and for every fuzzy sub modules A and B of X such that IA=IB, then A=B.

The following proposition characterizes fully cancellation fuzzy module in terms of its level modules.

Proposition 1.2.2:

Let be a fuzzy module of an R-module M, then X is fully cancellation fuzzy module if and only if X_t is fully cancellation module, ∀t∈(0,1].

Proof: (⟹)

Let K, N be two sub modules of R-module M. Let I: R ⟶[0, 1] such that:

$I(x) =\begin{cases} t \text{ if } x \in J \\ 0 \text{ otherwise} \end{cases}$, it is clear that I is a fuzzy ideal of R.

Let A: M ⟶[0, 1] =, B: ⟶[0, 1] such that;

$A(x) =\begin{cases} t \text{ if } x \in N \\ 0 \text{ otherwise} \end{cases} \forall\ t \in(0, 1]$

B(x) = $\begin{cases} t \text{ if } x \in K \\ 0 \text{ otherwise} \end{cases}$ ∀ t∈(0,1]

It is clear that A and B are fuzzy sub modules of X and A_t = N, B_t = K

$I_tA_t= I_tB_t \Longrightarrow (IA)_t=(IB)_t$[Definition (1.1.13)]∀ t∈(0,1] , so IA=IB . Hence A=B (since X is fully cancellation fuzzy module). Thus $A_t=B_t$. [By Definition 1.1.3. (3)]

Hence N=K ⟹ X_t=M is fully cancellation fuzzy module.

(⟸) If X_t is fully cancellation module to prove X is fully cancellation fuzzy module. Let A and B two fuzzy sub modules in X, let I be a fuzzy ideal of R such that IA=AB, hence $(IA)_t= (IB)_t$. That implies A_t , B_t are sub modules in X_t, for each t∈(0,1] ; since X_t is fully cancellation module , so $I_t A_t= I_t B_t$, implies that $A_t=B_t$. Hence A=B. Thus X fully cancellation fuzzy module.

Remarks and Examples 1.2.3:

Let X be a Fully Cancellation Fuzzy Module of the Z-Module Z.

Let I :nZ⟶[0,1] define by I(x)= $\begin{cases} t \text{ if } x \in nZ \\ 0 \text{ otherwise} \end{cases}$ for each t∈(0,1].

Let A: mZ⟶[0,1] , m ∈ Z such that ; A(x)= $\begin{cases} t \text{ if } x \in mZ \\ 0 \text{ otherwise} \end{cases}$ ∀ t∈(0,1]

Let B: sZ ⟶[0,1] , s ∈ Z such that ; B(x)= $\begin{cases} t \text{ if } x \in sZ \\ 0 \text{ otherwise} \end{cases}$ ∀ t∈(0,1]

It is clear that I is a fuzzy ideal of R and A, B are fuzzy sub modules of X.

Suppose that IA=IB if and only if $I_t A_t= I_t B_t$ and I_t=nZ, A_t=mZ, B_t=sZ, so

nZ. mZ=nZ. sZ, then <$\overline{nm}$> = <$\overline{ns}$> which implies that nm=nsa and ns=nmb

For some a, b∈Z Therefore nm=nmba, then either a=b=1 or a=b= −1. In each

Case we get nm=ns, so m=s. But m, s ∈ Z, then mZ=sZ, implies that $A_t=B_t$,

Hence A=B. Thus X is a fully cancellation fuzzy module. (By proposition 1.2.2)

We gives this Example to Show X is Not Fully Cancellation Fuzzy Module

Let M=Z_4 is a Z-module of a ring R. Let X: Z_4 ⟶[0, 1] define by X (t) = $\begin{cases} 1 \text{ if } x \in Z_4 \\ 0 \text{ other wise} \end{cases}$.

It is clear that X is a fuzzy module of a Z-module Z_4.

Define A: $(\bar{2})$ ⟶[0, 1] such that A(x) = $\begin{cases} t \text{ if } x \in (\bar{2}) \\ 0 \text{ otherwise} \end{cases}$ ∀ t∈(0,1]

Define B: Z_4 ⟶[0, 1] such that B(x) = $\begin{cases} t \text{ if } x \in Z_4 \\ 0 \text{ otherwise} \end{cases}$ ∀ t∈(0,1]

Define I: $(\bar{4})$ ⟶[0, 1] such that I(x) = $\begin{cases} t \text{ if } x \in (\bar{4}) \\ 0 \text{ otherwise} \end{cases}$ ∀ t∈(0,1]

It is clear that A and B are fuzzy sub modules of X and I is a fuzzy ideal of R M=X_t = Z_4 is not fully cancellation module [2, Remark and Examples (2.3) (2)]. Implies that X is not fully cancellation fuzzy module since if we take I_t= 4Z,

$A_t= (\bar{2})$ and $B_t=Z_4$. It is clear that $I_t A_t= I_t B_t$ since $(4Z)\ (\bar{2})= (4Z)\ (Z_4) = (\bar{0})$ but $Z_4\neq (\bar{2})$ and by Proposition (1.2.2) A≠B. Thus X is not fully cancellation fuzzy module.

Any Fuzzy Sub Module of a Fully Cancellation Fuzzy Module is a Fully Cancellation Fuzzy Module

Proof

Let X be a fully cancellation fuzzy module. Let C be a fuzzy sub module of a fully cancellation module X. Let $O_1\neq I$ be a fuzzy ideal of a ring R. Let A and B are fuzzy sub modules of C. Let IA=IB. To prove A = B?

Since IA=IB and A, B are fuzzy sub modules of fuzzy module C, and C is a fuzzy sub module of X. But X is a fully cancellation fuzzy module =, then A=B. Thus C is a fully cancellation fuzzy module.

Let X_1 be a Fully Cancellation Fuzzy Module of R-Module M_1 , and let X_2 be a Fuzzy Module of R-Module M_2 and $M_1\cong M_2$ if $X_1\cong X_2$, then X_2 is a Fully Cancellation Fuzzy Module

Proof:

Let X_1: $M_1 \longrightarrow [0, 1]$ define by $X_1(x) =\begin{cases} 1 \text{ if } x \in M_1 \\ 0 \text{ otherwise} \end{cases}$.

Let X_2: $M_2 \longrightarrow [0, 1]$ define by $X_2(y) =\begin{cases} 1 \text{ if } y \in M_2 \\ 0 \text{ otherwise} \end{cases}$

It is clear that X_1 and X_2 are fuzzy modules of M_1 and M_2 respectively.

Since $X_1\cong X_2$ and X_1 is a fully cancellation fuzzy module. Then X_2 is a fully cancellation fuzzy module. Since $(X_1)_t= M_1$ and $(X_2)_t= M_2$ for each $t\in(0,1]$ and $M_1\cong M_2$, M_1 is fully cancellation module. Then M_2 is fully cancellation module. [2, Remark and Examples (2.3) (6)] Thus $(X_2)_t= M_2$ is fully cancellation fuzzy module . (By Proposition1.2.2) Therefore X_2 is fully cancellation fuzzy module.

The Homomorphic Image of a fully Cancellation Fuzzy Module is not Necessary be a Fully Cancellation Fuzzy Module, the following Example to Show That

Letπ: $Z\longrightarrow Z/Z_4\cong Z_4$ be a natural epimorphism.

Define X: $Z \longrightarrow [0, 1]$, Y_1: $Z_4 \longrightarrow [0, 1]$ such that:

$X(x) =\begin{cases} 1 \text{ if } x \in Z \\ 0 \text{ otherwise} \end{cases}$, $Y(x) =\begin{cases} 1 \text{ if } Y \in Z_4 \\ 0 \text{ otherwise} \end{cases}$. It is easy to show that X and Y are fuzzy modules and $X_t=Z$, $Y_t =Z_4$ for each $t \in(0, 1]$, X is fully cancellation fuzzy module since $X_t=Z$ is fully cancellation module . But Y is not fully cancellation fuzzy module since for each $t\in(0,1]$ $Y_t=Z_4$ is not fully cancellation module (by Remark 1.2.3(2)).

Definition 1.2.4:

Let I be a fuzzy ideal of a ring R, I is called a cancellation fuzzy ideal if AI = BI where A and B are fuzzy ideal of R, then A=B. [12, Definition (2.2)]

Definition 1.2.5:

A fuzzy module X of an R-module M is called fuzzy simple if and only if X has no fuzzy proper sub module (in fact X is fuzzy simple if and only if $X=0_1$). [14, Definition (1.2.5)]

Definition1.2.6:

Let X is a fuzzy module on an R-module M. Then X is said to be faithful if F-ann$X=0_1$ where; F-ann$X=\{x_t$: $r_\ell x_t=0_1$ for all $x_t\subseteq X$ and r_ℓ be a fuzzy singleton of R, $\forall t,\ell\in(0,1]$ } . [15, Definition (3.2.6)]

Recall that if A and B are two fuzzy sub modules of a fuzzy module X, such that $A\subseteq B$. Then F-ann$B\subseteq$F-annA. [14, Remark (1.3.6)]

Remark1.2.7:

Let X is a fuzzy module on an R-module M. If X is fully cancellation fuzzy module which is not fuzzy simple, then X is fuzzy faithful module.

Proof:

Let $r_\ell\subseteq$F-annX, where r_ℓ be a fuzzy singleton of R $\forall\ell\in(0,1]$

Suppose that $r_\ell\neq 0_1$, then $r_\ell X=0_1$, and let A be a proper fuzzy sub module of X.

Hence $r_\ell A=0_1$ by [14, Remark (1.3.6)] .Thus $r_\ell X= r_\ell A$ and this implies X=A. Which is contradiction?

Definition1.2.8:

A fuzzy ideal I of a ring R is called a principle fuzzy ideal if there exists $x_t\subseteq I$ such that $I=(x_t)$ for each $m_s\subseteq I$, there exists a fuzzy singleton of R such that $m_s=a_\ell x_t$ where $s,\ell,t \in[0,1]$, that is $I=(x_t) =\{ m_s\subseteq I\backslash\ m_s=a_\ell x_t$ for some fuzzy singleton a_ℓ of R}. [7]

Now, we give this proposition.

Proposition 1.2.9:

Let R be a fuzzy principle ideal domain and let X be a fuzzy faithful fully cancellation fuzzy module of an R-module M and $X\neq 0_1$. Then X is not fuzzy simple.

Proof:

Suppose that X is fuzzy simple. Then X has only two fuzzy sub modules (0_1) and X.

Now, X is fuzzy faithful which implies that F-ann$X=0_1$, then $r_\ell X=0_1$,where r_ℓ be a fuzzy singleton of R $\forall\ \ell\in(0,1]$, hence $r_\ell X=r_\ell 0_1 \forall\ \ell\in(0,1]$, that is $(r_\ell)X=(r_\ell)(0_1)$. But X is fully cancellation fuzzy module, then $X=0_1$ which is contradiction!

Thus X is not fuzzy simple module.

The following is a characterization of fully cancellation fuzzy modules.

Theorem1.2.10:

Let X be a fuzzy module on an R-module M , let A and B be two fuzzy sub modules of X , let I be a non- empty fuzzy ideal of R , then the following statements are equivalent :-

1. X is a fully cancellation fuzzy module.
2. If $IA\subseteq IB$, then $A\subseteq B$

3. If I (x_t)⊆IB , then x_t⊆B where x_t⊆X , ∀ t∈(0,1] .

Proof:

(1)⟹(2) Since IA⊆IB , then IB= IA+IB=I(A+B)[1,p.16] , and since X is a fully cancellation fuzzy module , then B=A+B and this means A⊆B.

(2)⟹(3) Since I(x_t)⊆B, then by (2) x_t⊆B ∀ t∈(0,1].

(3)⟹(1) Let I(x_t)⊆B and x_t⊆B ∀ t∈(0,1] . To show that X is a fully cancellation fuzzy module

Let IA=IB to prove A=B?

Let x_t⊆A, then I (x_t)⊆IA⊆IB and by (3) x_t⊆B ∀ t∈(0,1] .

Thus A⊆B

To prove B⊆A?

Let x_t⊆B, I (x_t) ⊆IB and x_t⊆A ∀ t∈(0,1] . Then B⊆A

Therefore A=B.

As an immediate consequence of proposition (1.2.10) we have:-

Proposition 1.2.11:

Let X is a fuzzy module on an R-module M. Then X is fully cancellation fuzzy module if and only if (A: $_R$B) = (IA: $_R$IB) for all A and B are fuzzy sub modules of X and I is a fuzzy ideal of R.

(⟹)Let X be a fully cancellation fuzzy module. To prove (A: $_R$B) = (IA: $_R$IB)?

i. e. To prove (i) (IA: $_R$IB)⊆ (A: $_R$B)?

(ii) (A: $_R$B)⊆(IA:$_R$IB) ?

(i) Let a_ℓ⊆(IA:$_R$IB) , then a_ℓ. IB⊆IA ∀ ℓ ∈(0, 1], so I. a_ℓ. B⊆IA. Thus a_ℓ. B⊆A by (theorem 1.2.10 (3). Then a_ℓ ⊆(A:$_R$B) ∀ ℓ∈(0,1] . Thus (IA:$_R$IB)⊆ (A:$_R$B)

(ii) Let x_t⊆(A:$_R$B) ∀t∈(0,1] , so x_t. B⊆A , then I(x_t).B⊆IA , implies that (x_t)IB⊆IA

Therefore (x_t) ⊆(IA:IB) . Thus (A: $_R$B)⊆(IA:$_R$IB) . Hence (IA: $_R$IB) = (A: $_R$B)

(⟸) Let IA=IB. To prove A=B?

Now, (IA: $_R$IB) = (A: $_R$B), from the left side (IA: $_R$IB = λ_R (since IA=IB and by [2, p.5], where $\lambda_R(x)$ =1 ∀x∈R. [4]

Then (A: $_R$B) = λ_R and hence B⊆A.

Similarly (IB: $_R$IA) = (B: $_R$A). And IA=IB ⟹(IB: $_R$IA) =λ_R = (B: $_R$A), then λ_R= (B: $_R$A)

⟹ λ_RA⊆B⟹1.A⊆B⟹ A⊆B

Thus A=B.

Therefore X is fully cancellation fuzzy module.

Proposition 1.2.12:

Let X be a fully cancellation fuzzy module of an R-module M. If X is a cancellation fuzzy module, then every non-empty fuzzy ideal of R is a non-empty cancellation fuzzy ideal.

Proof:

Let X be a fully cancellation fuzzy module, let q, p be two fuzzy ideals of R such that Iq=Ip where I a non-empty fuzzy ideal of R.

Now, since Iq=Ip, then IqX=IpX. Since X is fully cancellation fuzzy module, then qX=pX. But X is a cancellation fuzzy module, hence q=p.

Thus I is a cancellation fuzzy ideal. [12, Definition (2.2)]

Recall that an element x in an R-module M is called a torsion element if rx = 0 for some non-zero divisor element r∈R [17].

Now, we shall fuzzify this concept as follows:-

Definition 1.2.13:

A fuzzy module X of an R-module M is called fuzzy torsion sub module if and only if for each $x_t \subseteq X$ there exist a fuzzy singleton r_ℓ of R , $r_\ell \neq 0$ such that $r_\ell x_t = 0_1 \forall\ t, \ell \in (0,1]$.

Proposition 1.2.14:

Let X be a fuzzy module over a principle fuzzy ideal I of R such that for all fuzzy singleton $x_t \subseteq X$ in non fuzzy torsion. Then X is fully cancellation fuzzy module.

Proof:

Let X be a fuzzy module over a principle fuzzy ideal I of R. To prove X is fully cancellation fuzzy module?

Let IA=IB where A, B are fuzzy sub module of X.

Since I is a principle fuzzy ideal, then I= (r_ℓ) for some fuzzy singleton r_ℓ of I and $r_\ell \neq 0_1$, then r_ℓ is a fuzzy singleton of R, $\forall \ell \in (0, 1]$. Implies that (r_ℓ) A= (r_ℓ) B, then we have $r_\ell a_t = (r_\ell)$ B for any $a_t \subseteq A$ $\forall t \in (0, 1]$. Thus $r_\ell a_t = r_\ell b_s$ for some bs⊆B.

$(ra)_\lambda = (rb)_\lambda$ where $\lambda = \min\{ t, \ell, s\}$ $\forall s \in (0,1]$.

$ra = rb \Longrightarrow ra - rb = 0 \Longrightarrow (ra - rb)_\lambda = 0_1 \Longrightarrow r_\ell a_t - r_\ell b_s = 0_1 \Longrightarrow r_\ell(a_t - b_s) = 0_1$

If $a_t - b_s \neq 0_1$, then $r_\ell \neq 0_1$ which is a contradiction since $a_t - b_s \subseteq X$ and every fuzzy singleton of X is non fuzzy torsion.

Thus $a_t - b_s = 0_1$, implies that $(a_t - b_s)_\lambda = 0_\lambda$ where $\lambda = \min\{t, \ell, s\}$

$a - b = 0 \Longrightarrow a = b \Longrightarrow a_t = b_s$ for all $a_t \subseteq A$ and $b_s \subseteq B$

Thus A=B .therefore X is a fully cancellation fuzzy module.

Definition 1.2.15:

A fuzzy singleton $0\neq a_t$ of a fuzzy ring is said to be a fuzzy zero divisor if there exist a fuzzy singleton b_ℓ of R such that $a_t.\ b_\ell=0_t$ where $b_\ell\neq 0_1$ and $t, \ell \in (0, 1]$. [13]

Definition 1.2.16:

A fuzz ring R is said to be a fuzzy integral domain if R has no zero divisor. [13]

Before we give our corollary, the following definition and lemma are needed.

Definition 1.2.17:

A fuzzy module X of an R –module M is called fuzzy cyclic module if there exists $x_t \subseteq X$ such that $y_k \subseteq X$ written as $y_k = r_t x_t$for some fuzzy singleton of R where k,, ℓ t$\in (0,1]$ in this case, we shall write X= (x_t) to denote the fuzzy cyclic module generated by x_t. [14, Definition (1.3.7)]

Lemma 1.2.18:

Let X is a fuzzy module over a fuzzy integral domain R. If X is cyclic fuzzy module generates by a non fuzzy torsion fuzzy singleton $x_t \subseteq X$. Then for all every non-empty fuzzy singleton $x_t \subseteq X$ is a non fuzzy torsion.

Proof:

Let $x_t \subseteq X$ where x_t be a fuzzy singleton of a fuzzy module X, $x_t \neq 0_1 \forall t \in (0, 1]$.

Now, suppose that x_t is a fuzzy torsion. Then there exist r_ℓ a fuzzy singleton of R and $r_\ell \neq 0_1 \forall \ell \in (0, 1]$ such that $r_\ell x_t = 0_1$. But $x_t \subseteq X$ and X is fuzzy cyclic module such that $x_t \subseteq (b_s)$, $b_s \subseteq X$.

Thus $x_t = a_k b_s \forall k, s \in (0, 1]$ for some fuzzy singleton a_k of R.

Now, $r_\ell x_t = r_\ell a_k b_s = 0_1$. But b_s is a non fuzzy torsion by hypothesis. Thus $r_\ell a_k = 0_1$, and R a fuzzy integral domain and $r_\ell \neq 0$, then $a_k = 0_1$

Therefore $a_k b_s = x_t = 0_1$ which is a contradiction!

Corollary1.2.19:

Let X is a fuzzy module over a fuzzy principle ideal of R. If X is cyclic fuzzy module generated by a non fuzzy torsion element, then X is fully cancellation fuzzy module.

Proof:

By Lemma (1.2.18) and proposition (1.2.14) we get the proof.

3 .NATURALLY CANCELLATION FUZZY MODULE

Naturally cancellation module is introduce by using the naturally product of sub modules which introduced in [2] where for each sub modules N and K of M, the naturally product of N and K (denoted by N.K) is define by (N: $_R$M) (K: $_R$M) M .An R-module M is called naturally cancellation module if for each sub modules N, K, W of M such that NW = KW implies N = K [2, Definition (2.2)]

The purpose of this section is to fuzzify this concept to naturally cancellation fuzzy modules. Many properties and results are introduced.

First, we start with the following definition.

Definition1.3.1:

Let X be a fuzzy module of an R-module M. X is called naturally cancellation fuzzy module if whenever A, B_1 and B_2 are fuzzy sub modules of X such that

$AB_1=AB_2$ then $B_1=B_2$.

The following proposition characterizes naturally cancellation fuzzy module in terms of its level modules

Proposition1.3.2:

Let X is a fuzzy module of an R-module M, then X is naturally cancellation fuzzy module $\Leftrightarrow X_t$ is naturally cancellation module.

Proof: ($\Longrightarrow$)

Let A: M$\longrightarrow [0,1]$ such that A(x) =$\begin{cases} t \text{ if } x \in N \\ 0 \text{ otherewise} \end{cases}$, where N is a sub module of M.

B: M$\longrightarrow$ [0,1] such that B(x) =$\begin{cases} t \text{ if } x \in k \\ 0 \text{ otherewise} \end{cases}$, where K is a sub module of M

C: M$\longrightarrow$ [0,1] such that C(x) =$\begin{cases} t \text{ if } x \in s \\ 0 \text{ otherewise} \end{cases}$, where S is a sub module of M

It is clear that A, B and C are fuzzy sub modules of X and A_t=N and B_t=K and C_t=S

Hence $A_t B_t = A_t C_t \Longrightarrow (AB)_t = (AC)_t, \forall t \in (0,1]$

AB=AC $\Longrightarrow$B=C (since X is naturally cancellation fuzzy module).

Then $B_{T=}C_t$. Hence X_t is a naturally cancellation fuzzy module.

($\Longleftarrow$) If X_t is a naturally cancellation module to prove X is naturally cancellation fuzzy module?

Let A, B and C is fuzzy sub modules of X. Such that AB=AC, hence $(AB)_t=(AC)_t, \forall t \in (0,1]$.

Therefore $A_t B_t = A_t C_t$ [3].and A_t, B_t, C_t are sub modules of X_t, $\forall t \in (0,1]$, but X_t is naturally cancellation module. Then $B_{T=}C_t$. therefore B= C

Thus X is naturally cancellation fuzzy module.

Definition1.3.3:

A fuzzy module X of an R –module M is called fuzzy multiplication module if for each non-empty fuzzy sub modules A of X there exists a fuzzy ideal I of R such that A=IX. [14, Definition (2.2.1)]

Now, we give this Definition.

Definition1.3.4:

Let X be fuzzy module of an R-module M. X is called fuzzy P_s- torsion if for every $x_t \subseteq A$, where A is a fuzzy sub module of X and$\forall t \in (0,1]$. There exsits a fuzzy singleton$r_\ell$of a fuzzy maximal ideal p of R. $\forall \ell, t \in (0,1]$such that $(\lambda_R - r_\ell)$ $x_t=0_1$

Where $(\lambda_R(x) = 1)$ for all x∈R. [4]

Now, we give this lemma to show that the Definition (1.3.4) implies the definition (1.3.3)

Lemma1.3.5:

If X is a fuzzy module of an R-module M. and X is a fuzzy P_s- torsion then X is a fuzzy multiplication.

Proof;

Let A be a fuzzy sub module of X and $x_t \subseteq A$

Now $(\lambda_R - r_\ell)$ $x_t=0_1$, Where $\lambda_R(x) = 1, \forall x \in R$

$\Rightarrow (1-r_\ell)\ x_t=0_1 \Rightarrow (x_t - r_\ell x_t) = 0_1 \Rightarrow (x_t-(rx))_t=0_1$ where $t=\min\{\ell, t\}$

$\Rightarrow (x- rx)_t=0_1 \Rightarrow (x- rx)_t=0_t \Rightarrow x- rx=0$ where $t=\min\{1, t\}$[if $A=B \Leftrightarrow A_t=B_t$]

$x=rx \Rightarrow x_t= (rx)_t \Rightarrow x_t= r_\ell x_t \Rightarrow A=PX$ since $x_t \subseteq A \subseteq X$

Therefore X is a fuzzy multiplication.

We introduced this proposition is needed in proposition. (1.3.7)

Proposition 1.3.6:

A fuzzy module X of an R-module M is multiplication if and only if every non-empty fuzzy sub module A of X such that $A= (A:_R X)\ X$.

Proof:

(⟹) Since X is a fuzzy multiplication module.

Then every a non-empty fuzzy sub module A of X. is written by A=IX for every fuzzy ideal I of a ring R.

To show that $A= (A:_R X)\ X$.

Let r_ℓ be a fuzzy singleton of R such that $r_\ell \subseteq I, \forall \ell \in (0,1]$

Implies that $r_\ell.X \subseteq IX \Rightarrow r_\ell.X \subseteq A \Rightarrow r_\ell \subseteq (A:_R X) \Rightarrow I \subseteq (A:_R X)$ and $A=IX \subseteq (A:_R X)X$

Thus $A \subseteq (A:_R X)\ X$

(⟸) To show that $(A:_R X)\ X \subseteq A$?

Let $r_\ell \subseteq (A:_R X) \Rightarrow r_\ell.X \subseteq A \Rightarrow r_\ell.X \subseteq IX$ (since X fuzzy multiplication module)$\Rightarrow r_\ell \subseteq I$,implies that $(A:_R X) \subseteq I \Rightarrow (A:_R X)X \subseteq IX \Rightarrow (A:_R X)X \subseteq A$

Thus $A= (A:_R X)\ X$.

Proposition1.3.7:

Let X is a multiplication cancellation fuzzy module. Then X is naturally cancellation fuzzy module if every fuzzy ideal of R is a fuzzy cancellation.

Proof:

Let X is a multiplication cancellation fuzzy module. Let A, B and C are fuzzy sub module of X such that A.B=A.C.

To prove B=C?

$\Longrightarrow$A.B=$(A:_RX)(B:_RX)X=(A:_RX)(C:_RX)X$=A.C

But X is cancellation fuzzy module, implies that $(A:_RX)(B:_RX)=(A:_RX)(C:_RX)$[2].

But $(A:_RX)$ be a fuzzy ideal (by Definition 1.1.21)

Thus $(B:_RX)=(C:_RX)$. [2].

But X is a multiplication fuzzy module.

Thus $(B:_RX)X=(C:_RX)X$ by (proposition 1.3.6)

Hence B=C which that's

X is a naturally cancellation fuzzy module.

Definition1.3.8:

Let X is a fuzzy module of an R-module M .X is called a finitely generated fuzzy module if there exists $x_1, x_2, x_3 \ldots\ldots \subseteq X$ such that $X=\{a_1(x_1)t_1+a_2(x_2)t_2+\ldots\ldots..+a_n(x_n)\ t_n$, where $a_i \in R$ and $a(x)_t=(ax)_t$, $\forall t \in (0,1]$Where $(ax)_{t(y)}=\begin{cases} t \text{ if } y=ax \\ o\ otherwise \end{cases}$.[12, Definition (2.11)].

Proposition1.3.9:

If X is a finitely generated fuzzy module Then X_t is finitely generated module,$\forall t \in (0,1]$ [12, proposition (2.12)]

The following lemma is used in the next corollary.

Lemma1.3.10:

Let X be a fuzzy faithful module of an R-module M. if and only if X_t is faithful module.

Proof:

($\Longrightarrow$) let X be a fuzzy faithful module. To prove X_t is a faithful module$\forall t \in (0,1]$.

We claim that annX_t =0$\forall t \in (0,1]$.

Let r $\in$annX_t. To show that r=0, implies that rx=0 for all r$\in$R, x$\in X_t$

Now, $(rx)_t=r_t.x_t=0_t\leq 0_1,\forall t \in (0,1]$ Then $r_t\subseteq$F-annX for all $x_t\subseteq X$

But F-annX=o_1, then $r_t=0_t\subseteq 0_1,\forall t \in (0,1]$

This implies that r=0. (By Definition 1.1.3(3))

Therefore X_t is faithful module.

(⟸) conversely, X_t is faithful module. To show that X is fuzzy faithful module

Let $r_k \subseteq$F-annX. To prove $r_k=0_1$?

Now $r_k x_t=0_1, \forall x_t \subseteq X, \forall t, k \in (0,1]$.

$\Rightarrow (rx)_\lambda=0_1$, if $1=\lambda \Rightarrow (rx)_1=0_1 \Leftrightarrow rx=0 \Rightarrow r\in annX_t \Rightarrow r=0$ (since X_t is faithful)

Then $r_k=0_k$

Which implies that $r_k \subseteq$F-annX?

Then X is fuzzy faithful module.

Corollary 1.3.11:

Let X is a finitely generated faithful multiplication fuzzy module of an R-module M. Then X is naturally cancellation fuzzy module if for every fuzzy ideal of R is a cancellation fuzzy ideal.

Proof:

Let X is finitely generated faithful multiplication fuzzy module. Then X_t=M is a finitely generated faithful multiplication (by Lemma 1.3.13)

Implies that X_t =M is cancellation R-module by [13, Theorem 3.1]

Thus X is cancellation fuzzy module by [12. Proposition 2.3]

But X is a multiplication fuzzy module

Therefore X is naturally cancellation fuzzy module by (Proposition 1.3.7)

4. FULLY CANCELLATION AND NATURALLY CANCELLATION FUZZY MODULE

As we have mentioned in section two and three we study the concept fully cancellation and naturally cancellation fuzzy module and show that this concept is not equivalent in general.

In this section, we will show that in the class of fuzzy multiplication modules the two concepts of fully and naturally cancellation fuzzy module is equivalents. Moreover, we will show that in the class of fuzzy cyclic modules the concept in also equivalent.

Now, we give this remark.

Remark: 1. 4.1

Let X be a fully cancellation fuzzy module of an R-module M. then not necessary X is naturally cancellation fuzzy module. This example we show that

Example1.4.2:

Let X: Q⟶ $[0,1]$ defined by $X(x)=\begin{cases} 1 \text{ if } x\in Q \\ 0 \text{ otherewise} \end{cases}$

It is clear that X is a fuzzy module of Q.

Let A:$\frac{1}{4}$ Z⟶ $[0,1]$ defined by a (x) =$\begin{cases} t \text{ if } x \in \frac{1}{4} Z \\ 0 \text{ otherewise} \end{cases}$ $\forall t \in (0,1]$

It is clear that A is a fuzzy sub module of Z.

Let B:$\frac{1}{2}$ Z⟶ $[0,1]$ defined by B(x) = $\begin{cases} t \text{ if } x \in \frac{1}{2} Z \\ 0 \text{ otherewise} \end{cases}$ $\forall t \in (0,1]$

It is clear that B is a fuzzy sub module of Z.

Let C: Z⟶ $[0,1]$ defined by C(x) =$\begin{cases} t \text{ if } x \in Z \\ 0 \text{ otherewise} \end{cases}$ $\forall t \in (0,1]$

It is clear that C is a fuzzy sub module of Z.

X_t =Q is not a naturally cancellation module.

Since $A_t = \frac{1}{4} Z$, $B_t = \frac{1}{2} Z$, $C_t = Z$, $\forall t \in (0,1]$

$A_t. B_T = (\frac{1}{4} Z:_R Q) (\frac{1}{2} Z:_R Q) Q = 0$, $A_t. C_t = (\frac{1}{4} Z:_R Q) (Z:_R Q) Q = 0$.

Then $A_t. B_t = A_t. C_t$.

But $B_t \neq C_t$, since$\frac{1}{4} Z \neq Z$.Then B≠C (by proposition (1.3.2))

Thus Q is not naturally cancellation fuzzy module.

To show that X is fully cancellation fuzzy module

Let I: Nz⟶ $[0,1]$ such that I(x) =$\begin{cases} t \text{ if } x \in nZ \\ 0 \text{ otherewise} \end{cases}$ where n∈ $Z \forall t \in (0,1]$

It is clear that I is a fuzzy ideal of R

If IA=IB, then to prove A=B?

Since A and B are fuzzy sub modules of X of an Z module Q, then $x_t \subseteq A \Longrightarrow A(x) \geq t$, x_t is a fuzzy singleton of A.

$\Longrightarrow x \in A_t \Longrightarrow nZ = I_t \Longrightarrow n \in I_t$, $\forall t \in (0,1]$

$\Longrightarrow Nx \in I_t A_t = I_t B_t$, for some y∈B =, thus $x = y \in B_t$

Therefore $A_t \subseteq B_t$

With the same mouthed we prove that $B_t \subseteq A$, thus $A_t = B_t$, implies that A=B (by Definition 1.1.3(3)).

Hence X is fully cancellation fuzzy module.

The concept fully cancellation fuzzy module is equivalent to a concept naturally cancellation fuzzy module under the condition where X is fuzzy multiplication.

The following theorem gives the relationship between fully cancellation and naturally cancellation fuzzy module under the condition fuzzy multiplication module.

Theorem1.4.3:

If X is a fuzzy multiplication module then X is naturally cancellation fuzzy module if and only if X is fully cancellation fuzzy module.

Proof:

($\Longrightarrow$) Let X is a fuzzy multiplication module of an R-module M.

Let I be a non-empty fuzzy ideal of a ring R.

Let A and B be two fuzzy sub modules of X such that IA=IB. To prove A=B?

Since X is a fuzzy multiplication module, then A= JX and J= $(A:_R X)$.

Then $IA=I(A:_R X)X=(A:_R X)IX=(A:_R X)(IX:_R X)X=AIX$, and

$IB=I(B:_R X)X=(B:_R X)IX=(B:_R X)(IX:_R X)X=BIX$.

Therefore AIX=BIX.

But X is naturally cancellation fuzzy module, thus A=B.

Therefore X is fully cancellation fuzzy module.

($\Longleftarrow$) Let X is a fully cancellation fuzzy module.

To show that X is a naturally cancellation fuzzy module

Let A, B and C are fuzzy sub modules of a fuzzy module X, such that AB=AC.

To prove B=C?

Since X is a fuzzy multiplication module. then $A=(A:_R X)X, B=(B:_R X)X, C=(C:X)X$

$\Longrightarrow$AB=AC

$(A:_R X)(B:_R X)X=(A:_R X)(C:_R X)X$

$(A:_R X)B=(A:_R X)C$

Since X is a fully cancellation fuzzy module, then B=C

Therefore X is a naturally cancellation fuzzy module.

Corollary 1.4.4:

Let X be a fuzzy cyclic R-module then X is a naturally cancellation fuzzy module if and only if X is a fully cancellation fuzzy module.

The following theorem gives some characterization for fully and naturally cancellation fuzzy module.

Theorem 1.4.5:

Let X be a fuzzy multiplication module, let A, B and C are fuzzy sub modules of X and $x_t \subseteq X$, then the following statements are equivalents.

1. X is a naturally cancellation fuzzy module.
2. X is a fully cancellation fuzzy module.
3. If A.B ⊆A.C, where A, B and C are fuzzy sub modules of X, then B⊆C.
4. If A.(xt)⊆A.B, then xt⊆B, $\forall t \in (0,1]$
5. (A.B:$_R$A.C)= (B: $_R$C).

Proof:

(1)⟹(2)Since X be a fuzzy multiplication module, and X is a naturally

Cancellation fuzzy module then X is a fully cancellation fuzzy module. (By Theorem 1.4.3)

(2)⟹(3)Since X is a fuzzy multiplication module.

Then A= (A: $_R$X) X, B= (B: $_R$X) X, C=(C: $_R$X) X, and AB⊆AC

⟹(A: $_R$X)(B: $_R$X)X ⊆(A: $_R$X) (C: $_R$X) X ⟹(A: $_R$X) B ⊆(A: $_R$X) C.

Since X is a fully cancellation fuzzy module.

⟹B⊆C

(3)⟹(4)Since A.(x_t)⊆A.B, by(3), then x_t⊆B, ∀□ ∈ $(0,1]$, where x_t⊆X.

(2)⟹(5)Let x_t⊆(B:$_R$C), ∀□ ∈ $(0,1]$, where x_t⊆X.

x_t. C⊆ B, hence x_t(C: $_R$X) X ⊆ (B: $_R$X) X

Since X is fuzzy multiplication module. Then A=(A:$_R$X)X, B=(B:$_R$X)X, C=(C:$_R$X)X, and x_t(A:X)(C:X)X⊆(A:X)(B:X)X

⟹x_t (A.C)⊆A.B

Thus x_t⊆(A.B:$_R$A.C)

Therefore (B: $_R$C)⊆(A.B:$_R$A.C)

Let x_t⊆(AB:$_R$AC),∀ t ∈(01]

x_t AC⊆AB, hence x_t(A:$_R$X)C⊆(A:$_R$X)B

⟹x_t. C⊆B (since X is fully cancellation module)

Then X is fully cancellation fuzzy module, therefore x_t⊆(B:$_R$C)

Thus (AB: $_R$AC) ⊆(B: $_R$C). Then (AB: $_R$AC) = (B: $_R$C)

(5)⟹(2)Let A.B=A.C for A,B and C are fuzzy sub modules of X. To prove B=C?

Then (AB: RAC) =λ_R where$\lambda_R(x) = 1, \forall x \in R$ (By (5)): we get

(B: C)=λ_R and C⊆B.

Similarly (AB: $_R$AC) =λ_R=(C: $_R$B)

Then B⊆C

Therefore B=C.

(4) ⟹(2) it is clear.

Proposition 1.4.6:

Let X be a fully cancellation fuzzy module of an R-module M and let A be a fuzzy multiplication sub modules of X then A is a naturally cancellation fuzzy module.

Proof:

Since X is a fully cancellation fuzzy module and A is a fuzzy sub module of X

Then A is a fully cancellation fuzzy module (by Remark 1.2.3(3))

But A is a fuzzy multiplication module

Then A is naturally cancellation fuzzy module. (By Theorem.1.4.3)

As an immediate consequence of proposition (1.4.6) we have the following result:

Corollary 1.4.7:

Let X be a fully cancellation fuzzy module of an R-module M and if A is a fuzzy cyclic sub module of X. then A is a naturally cancellation fuzzy module.

5. DIRECT SUM OF FULLY CANCELLATION FUZZY MODULE

In this section, we introduce the concepts of external direct sum and internal direct sum of fully cancellation fuzzy module and we study some of their basic properties, namely, when the fuzzy modules are fully cancellation.

Definition 1.5.1:

Let X and Y are two fuzzy modules of M_1, M_2 respectively. Define $X\oplus Y: M_1\oplus M_2 \longrightarrow [0,1]$ by$(X\oplus Y)(a,b) = \min\{X(a), Y(b)$ for all $(a,b) \in M_1\oplus M_2\}$X⊕Y is called a fuzzy external direct sum of X and Y. [14, Definition (3.5.1)]

Definition 1.5.2:

If A and B are two fuzzy sub modules of X, Y respectively. Define $A\oplus B: M_1\oplus M_2 \longrightarrow [0,1]by(A\oplus B)(a,b) = \text{in}\{A(a), B(b)$for all $(a,b) \in M_1\oplus M_2\}$

Note that, if X=A+B and A∩B=0, then X is called the internal direct sum of A and B which is denoted byA⊕B. Moreover, A and B are called direct summands of X. [14, Definition (2.3.1)]

Remark 1.5.3:

X⊕YIs a fuzzy module of $M_1\oplus M_2$ by [14, Remark (3.5.2)]

Lemma 1.5.4:

If X and Y are fuzzy modules of M_1 and M_2 respectively

Then $X \oplus Y$ is a fuzzy module of $M_1 \oplus M_2$.[15, lemma (2.3.3)]

We introduce the following two lemma which is needed in our next result:-

Lemma 1.5.5:

Let X be a fuzzy module of an R-module M and $X = A_1 \oplus A_2$ where A_1 and A_2 are fuzzy submodules of X. If F-annA1+F-annA2 $=\lambda_R$ where $\lambda_R(x)=1, \forall x \in R$, then every fuzzy submodule B of X is written as $B = B_1 \oplus B_2$ where B_1, B_2 fuzzy submodules of A_1 and A_2 respectively .

Proof:

Let B any fuzzy sub module of X.

First we claim that $B = B_1 \oplus B_2$ for some fuzzy submodules B_1 of A_1 and B_2 of A_2

In fact $b_\ell \subseteq B \; \forall \ell \in (0,1]$, then $b_\ell = a_t + c_s \; \forall\, t, s \in (0,1]$ for some $a_t \subseteq A_1$ and $c_s \subseteq A_2$. Moreover, there exists, fuzzy singleton $L_r \subseteq F - annA_1$ and $K_i \subseteq F - annA_2, \forall\, r, i \in (0,1]$.

Such that Lr+Ki $=\lambda_R$, where $\lambda_{R(x)} = 1, \forall x \in R$

Now, let B1=$(F - annA_2).a_t$ and $a_t \subseteq A_1$, B2=$(F - annA_1).c_s$, and $c_s \subseteq A_2$.Then B1 is a fuzzy sub modules of A1and B2 is a fuzzy sub modules of A2

Now $a_t = \lambda_R.a_t$ where $\lambda_R(x) = 1, \forall x \in R$

$=1.a_t$. but Lr+ Ki =1=$\lambda_R, \forall x \in R$

(Lr+ Ki).$a_t = Lra_t + Kia_t = Kia_t \subseteq A_1$ and $c_s = \lambda_R.c_s$ =(Lr+ Ki).$c_s = Lrc_s + Kic_s$ =LrCs⊆A2

Then $b_\ell = a_t + c_s = Kia_t + Lrc_s \subseteq B_1 \oplus B_2$. Therefor $b_\ell \subseteq B_1 \oplus B_2$

For the other direction

Let Wj⊆ $B_1 \oplus B_2$, then Wj=$fma_t + dnc_s, \forall m, n. j \in (0, 1]$

For some fuzzy singleton fm⊆ $F - annA_2$ and dn⊆ $F - annA_1$

Wj=$fma_t + dnc_s = fma_t + dna_t + fmc_s + dnc_s$

=fm$(a_t + c_s)$ +dn$(a_t + c_s)$

Wj= $(fm + dn).b_\ell \subseteq B$ Thus $B = B_1 \oplus B_2$, where $a_t + c_s = b_\ell$

Lemma 1.5,6:

Let A and B be two fuzzy sub modules of a fuzzy module X such that $X = A \oplus B$. Then $X_t = A_t \oplus B_t$ for all t∈ *(0,1*]. [15, Lemma (2.3.3)]

The following proposition to show that fully cancellation fuzzy module under direct sum

Proposition 1.5.7:

Let X is a fuzzy module of an R-module M. And $X = A_1 \oplus A_2$ where A_1 and A_2 be two fuzzy sub modules of X. Such that F-annA1+F-annA2=λ_R where $\lambda_R(x) = 1, \forall x \in R$. Then A_1 and A_2 is fully cancellation fuzzy module if and only if

X is fully cancellation fuzzy module.

Proof:

(⟹) Let I be a non- empty fuzzy ideal of R, and let A and B be any two fuzzy sub modules of X. Let IA=IB to show that A=B?

Since F-annA1+F-annA2=$\lambda_R$where$\lambda_R(x) = 1, \forall x \in R$

By lemma A=$A_1 \oplus A_2$ and B=$B_1 \oplus B_2$, for some A_1, A_2 sub modules of A and for some B_1, B_2 sub modules of B. Thus I ($A_1 \oplus A_2$) =I ($B_1 \oplus B_2$). Hence $IA_1 \oplus IA_2 = IB_1 \oplus IB_2$, this implies that $IA_1 = IB_1$ and $IA_2 = IB_2$.But A_1 and A_2 are fully cancellation fuzzy modules. Then $A_1 = B_1$ and $A_2 = B_2$.

Hence A=B

(⟸) It is clear by (Remark 1.2.3(4))

Definition: 1.5.8:

Let f be a function from a set M in to a setM'. A fuzzy subset A of M is called f-invariant if A(x) =A(y), whenever f(x) =f(y) where x, y∈M. [13]

Definition: 1.5.9:

A fuzzy module X of an R-module M is called fully fuzzy invariant (fully f-invariant) if for every fuzzy sub module of X is an f-invariant.

Remark 1.5.10:

It is clear that every fully f-invariant module is f-invariant.

The following lemma which is needed to prove the proposition (1.5.12)

Lemma1.5.11:

Let X be a fuzzy module of an R-module M, and X=A ⊕B where A and B are two fuzzy submodules of X if C is fuzzy invariant fuzzy submodule of X, then C=(A∩C) ⊕(B∩C) .

Proof:

Let f: X⟶A and g: x⟶B be any epimorphism natural mapping.

Since C is a fuzzy invariant fuzzy sub module of X. Then C(x)=C(y) and f(x)=f(y),∀x,y ∈ M , C(x)=C(y) and g(x)=g(y),∀x,y ∈ M.

But f is epimorphism ∀x, y∈ M.

F (x)⊆A,∀x ∈ M,then f(c) ⊆ A and g(c)⊆B.

Then f(c)⊆ A∩C and g(c)⊆ B∩C

Now, C=I(c) =f(c) ⊕g(c)⊆(A∩C) ⊕(B∩C)

Where I be identity mapping

The other is direction of the inclusion is a obvious therefore C= (A∩C) ⊕(B∩C)

Proposition1.5.12:

Let X be a fuzzy module of an R-module M, and X=$A_1 \oplus A_2$ where A_1 and A_2 be two fuzzy sub modules of X such that A_1 and A_2 are fully invariant fuzzy sub modules. Then A_1, A_2 are fully cancellation fuzzy module if and only if X is a fully cancellation fuzzy module.

Proof:

(⇒) Let A, B are fuzzy sub modules of X and let I be anon-empty fuzzy ideal of R.

Suppose that IA=IB. To prove A=B?

Since A_1, A_2 are fully invariant fuzzy sub modules, then A= $(A \cap A_1) \oplus (A \cap A_2)$ and

B= $(B \cap A_1) \oplus (B \cap A_2)$. (By Lemma 1.5.11)

Therefore $I(A \cap A_1) \oplus (A \cap A_2) = I(B \cap A_1) \oplus (B \cap A_2)$ so $I(A \cap A_1) = I(B \cap A_1)$ and $I(A \cap A_2) = I(B \cap A_2)$.

Hence $A \cap A_1 = B \cap A_1$ and $A \cap A_2 = B \cap A_2$, since A_1, A_2 are fully cancellation fuzzy modules.

Thus A=B

That implies X is a fully cancellation fuzzy module.

(⇐) by Remark 1.2.3(4)

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